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The Lagrange multiplier rule for super efficiency in vector optimization ☆

H. Huang

Department of Mathematics, Yunnan University, Kunming 650091, PR China

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Abstract

In this paper, we study constrained multiobjective optimization problems with objectives being closed-graph multifunctions in Banach spaces. In terms of the coderivatives and Clarke's normal cones, we establish Lagrange multiplier rules for super efficiency as necessary or sufficient optimality conditions of the above problems.

Keywords: Vector optimization; Multifunction; Super efficient solution; Coderivative

1. Introduction

Let X be a Banach space, $f_i : X \to R \cup \{+\infty\}$ $(i = 1, 2, \ldots, n)$ be lower semicontinuous functions and $f_i : X \to R$ $(i = n+1, n+2, \ldots, m)$ be continuous functions. Many authors (see [8–10,18,19]) studied the following optimization problem with inequality and equality constraints:

$$\begin{aligned}
&\min f_0(x),\\
&f_i(x) \leqslant 0, \quad i = 1, \ldots, n,\\
&f_i(x) = 0, \quad i = n+1, \ldots, m,\\
&x \in \Omega. \qquad\qquad (1.1)
\end{aligned}$$

Under some restricted conditions (e.g., each f_i is locally Lipschitz), it is well known, as the Lagrange multiplier rule, that if $\bar{x}$ is a local solution of (1.1), then there exists $\lambda_i \in R$ $(0 \leqslant i \leqslant m)$ such that

$$0 \in \sum_{i=0}^{m} \lambda_i \partial f_i(\bar{x}) + N(\Omega, \bar{x}),$$

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E-mail address: huanghui@ynu.edu.cn.

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$$\sum_{i=0}^{m} |\lambda_i| = 1 \quad \text{and} \quad \lambda_i \geqslant 0,\ i = 0, 1, \dots, n,$$

where $\partial f_i(\bar{x})$ and $N(\Omega, \bar{x})$ denote the subdifferential and the normal cone (see Section 2 for their definitions). The main purpose of this paper is to establish the corresponding Lagrange multiplier rules for multifunctions in Banach spaces for super efficiency.

Let X, Y be Banach spaces, Ω be a closed subset of X (Ω can be regarded as an abstract geometric constraint) and $\Phi : X \to 2^Y$ be a multifunction. Let C be a closed convex pointed ($C \cap (-C) = \{0\}$) cone in Y, which specifies a partial order $\leqslant_C$ on Y as follows: for $y_1, y_2 \in Y$,

$$y_1 \leqslant_C y_2 \quad \text{if and only if} \quad y_2 - y_1 \in C.$$

Consider the following constrained multiobjective optimization problem

$$C - \min \Phi(x), \quad x \in \Omega. \tag{1.2}$$

Let A be a subset of Y and $a \in A$. Recall that a is called a Pareto efficient point of A, written as $a \in E(A, C)$, if

$$(A - a) \cap (-C) = \{0\}.$$

Following Borwein and Zhuang [2], we say that a is a super efficient point of A if there exists a real number $M > 0$ such that

$$\mathrm{cl}\big[\mathrm{cone}(A - a)\big] \cap (B_Y - C) \subset M B_Y,$$

where B_Y denotes the closed unit ball of Y. We use $SE(A, C)$ to denote the set of all super efficient points of A. It is known and easy to verify that $a \in SE(A, C)$ if and only if there exists a constant $M > 0$ such that

$$\|x - a\| \leqslant M\|y\| \quad \text{for all } x \in A \text{ and } y \in Y \text{ with } x - a \leqslant_C y.$$

It follows that $SE(A, C) \subset E(A, C)$. The super efficiency refines the notion of efficiency and other kinds of proper efficiency. Many authors [5,11–16] have studied the super efficiency. For $\bar{x} \in X$ and $\bar{y} \in \Phi(\bar{x})$, we say that $(\bar{x}, \bar{y})$ is a local super efficient solution of the multiobjective optimization problem (1.2) if there exists a neighborhood $U$of $\bar{x}$ such that

$$\bar{y} \in SE\big(\Phi(\Omega \cap U), C\big).$$

Recently, Borwein and Zhuang [1] established Lagrange multiplier theorem for super efficiency in convex setting. Rong and Wu [4] extended the results of Borwein and Zhuang [1], and presented Lagrange multipliers under cone-convexlike assumptions. Mehra [6] extended the results of Li [3] and Rong and Wu [4], and presented Lagrange multipliers under nearly convexlike assumptions. All the results mentioned above required that the objective function satisfies some kinds of convexity assumption and that the cone C has a bounded base. In 2006, Zheng and Ng [7] established Lagrange multiplier rules for Pareto efficiency by the coderivatives in Clarke's sense.

In this paper, dropping the assumption that the ordering cone has a bounded base, we give Lagrange multiplier rules for the problem (1.2) by the coderivatives. The results are the extensions of the necessary and sufficient conditions for local super efficient points to sets in [16] by Zheng, Yang and Teo. The rest of this paper is written as follows. In Section 2, we present some basic definitions and results that are required in sequel. Section 3 is devoted to the Lagrange multiplier rules. As application, we give Fermat's rule for super efficiency.

2. Preliminaries

In this section, we assume that X is a Banach space. Let $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $x \in \mathrm{dom}(f) := \{x \in X\colon\ f(x) < +\infty\}$, let $h \in X$, and let $f^0(x, h)$ denote the generalized directional derivative given by Rockafellar [10], that is,

$$f^0(x, h) := \lim_{\varepsilon \downarrow 0} \limsup_{z \xrightarrow{f} x,\, t \downarrow 0} \inf_{w \in h + \varepsilon B_X} \frac{f(z + tw) - f(z)}{t},$$

where the expression $z \xrightarrow{f} x$ means $z \to x$ and $f(z) \to f(x)$. It is known that $f^0(x,h)$ reduces to Clarke's directional derivative when f is locally Lipschitz (see [10]). Let

$$\partial f(x) := \{x^* \in X^*\colon\ \langle x^*, h\rangle \leqslant f^0(x,h) \text{ for all } h \in X\},$$

where X^* denotes the dual space of X. Let A be a closed subset of X and let $N(A,a)$ denote Clarke's normal cone of A at a, that is,

$$N(A,a) := \begin{cases} \partial\delta_A(a), & a \in A, \\ \emptyset, & a \notin A, \end{cases}$$

where δ_A denotes the indicator function of A: $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ otherwise. For $a \in A$, let $T(A,a)$ denote Clarke's tangent cone, namely,

$$T(A,a) := \{h \in X\colon\ d_A^0(a,h) = 0\},$$

where $d_A(\cdot)$ denotes the distance function to A. It is well known that for $a \in A$,

$$N(A,a) = \{x^* \in X^*\colon\ \langle x^*, h\rangle \leqslant 0 \text{ for all } h \in T(A,a)\}.$$

The following result [10, p. 52, Corollary] presents an important necessary optimality condition for a nonsmooth constrained optimization problem.

Proposition 2.1. *Let $f : X \to R$ be a locally Lipschitz function and A be a closed subset of X. Suppose that f attains its minimum over A at $a \in A$. Then* $0 \in \partial f(a) + N(A,a)$.

Recall that a vector v in X is said to be hypertangent to the set A at the point $a \in A$ if, for some $\varepsilon > 0$,

$$x + tw \in A, \quad \text{for all } x \in (a + \varepsilon B_X) \cap A,\ w \in v + \varepsilon B_X,\ t \in (0,\varepsilon).$$

We use $H(A,a)$ to denote the set of all vectors hypertangent to A at a [10, p. 57].

The following proposition [10, p. 105, Corollary] plays an important role in Theorems 3.1 and 3.2.

Proposition 2.2. *Let A_1 and A_2 be subsets of X and let $a \in A_1 \cap A_2$. Suppose that*

$$T(A_1,a) \cap H(A_2,a) \neq \emptyset.$$

Then

$$N(A_1 \cap A_2, a) \subset N(A_1,a) + N(A_2,a).$$

For $\Phi : X \to 2^Y$, a multifunction from X to another Banach space Y, let $\mathrm{Gr}(\Phi)$ denote the graph of Φ, that is,

$$\mathrm{Gr}(\Phi) := \{(x,y) \in X \times Y\colon\ y \in \Phi(x)\}.$$

We say that Φ is closed-graph if $\mathrm{Gr}(\Phi)$ is a closed subset of $X \times Y$, that Φ is convex-graph if $\mathrm{Gr}(\Phi)$ is a convex subset of $X \times Y$ and that Φ is locally convex-graph at $\bar{x}$ if $\mathrm{Gr}(\Phi) \cap ((\bar{x} + \delta B_X) \times Y)$ is a convex subset of $X \times Y$ for some $\delta > 0$. Recently, Zheng, Yang and Teo [20] introduced the following definition: a closed set A in X is said to be strongly normal at $a \in A$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\langle x^*, u - a\rangle \leqslant \varepsilon\|u - a\| \quad \text{for all } x^* \in N(A,a) \cap B_{X^*} \text{ and } u \in A \cap (a + \delta B_X).$$

It is natural for us to say that Φ is strongly normal at $(\bar{x},\bar{y}) \in \mathrm{Gr}(\Phi)$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle + \langle y^*, y - \bar{y}\rangle \leqslant \varepsilon(\|x - \bar{x}\| + \|y - \bar{y}\|)$$

for all $(x^*, y^*) \in N(\mathrm{Gr}(\Phi), (\bar{x},\bar{y})) \cap (B_{X^*} \times B_{Y^*})$ and all $x \in \bar{x} + \delta B_X$ and $y \in \Phi(x) \cap (\bar{y} + \delta B_Y)$. Motivated by this definition, we introduce the following definition.

Definition 2.1. We say that Φ is side-strongly normal at $(\bar{x},\bar{y}) \in \mathrm{Gr}(\Phi)$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle + \langle y^*, y - \bar{y}\rangle \leqslant \varepsilon\|y - \bar{y}\|$$

for all $(x^*, y^*) \in N(\mathrm{Gr}(\Phi), (\bar{x},\bar{y})) \cap (B_{X^*} \times B_{Y^*})$ and all $x \in \bar{x} + \delta B_X$ and $y \in \Phi(x)$.

Clearly,

$$\text{local convex-graph property} \quad \Rightarrow \quad \text{side-strong normality} \quad \Rightarrow \quad \text{strong normality},$$

but the converse is not true. See the following two examples.

Example 2.1. Let $X = Y = R$. Let $\phi(x) = |x| + x^5$ for all $x \in R$. This function is Lipschitz near 0, and it is easy to show that $\phi^0(0; v) = |v|$ for all $v \in R$ and $\partial\phi(0) = [-1, 1]$. Take $(\bar{x}, \bar{y}) = (0, 0)$. Let $\Phi(x) = [\phi(x), +\infty)$ for all $x \in R$. By [10, p. 61, Corollary],

$$N\big(\mathrm{Gr}(\Phi), (0,0)\big) = \big\{r\big(x^*, -1\big)\colon\ x^* \in \partial\phi(0),\ r \geqslant 0\big\} = \big\{r\big(x^*, -1\big)\colon\ x^* \in [-1, 1],\ r \geqslant 0\big\}.$$

Given $\varepsilon > 0$. Take $\delta = \sqrt[4]{\frac{\varepsilon}{1+\varepsilon}}$. Then for all $x \in 0 + \delta B_X = [-\delta, \delta]$, $y = |x| + x^5 + h \in \Phi(x)$ $(h \geqslant 0)$ and $r(x^*, -1) \in N(\mathrm{Gr}(\Phi), (0,0)) \cap (B_{X^*} \times B_{Y^*}) = \{r(x^*, -1)\colon\ x^* \in [-1, 1], 0 \leqslant r \leqslant 1\}$, we have

$$\big\langle rx^*, x - 0\big\rangle + \langle -r,\ y - 0\rangle = r\big(x^*x - |x| - x^5 - h\big) \leqslant r\big(|x| - |x| - x^5 - h\big) \leqslant r\big(-x^5\big) \leqslant |x|^5.$$

Noting that when $x \neq 0$,

$$\begin{aligned} |x| \leqslant \delta \quad &\Rightarrow \quad |x|^4 \leqslant \frac{\varepsilon}{1+\varepsilon} \quad \Rightarrow \quad 1 + \frac{1}{\varepsilon} \leqslant \frac{1}{|x|^4} \quad \Rightarrow \quad \frac{1}{\varepsilon} \leqslant \frac{1}{|x|^4} - 1 \quad \Rightarrow \quad 1 \leqslant \varepsilon\left(\frac{1}{|x|^4} - 1\right) \\ &\Rightarrow \quad |x|^5 \leqslant \varepsilon\big(|x| - |x|^5\big) \quad \Rightarrow \quad |x|^5 \leqslant \varepsilon\big(|x| + x^5 + h\big), \end{aligned}$$

it follows that

$$\big\langle rx^*, x - 0\big\rangle + \langle -r, y - 0\rangle \leqslant \varepsilon\big(|x| + x^5 + h\big) = \varepsilon|y - 0| \quad \text{for all } x \in [-\delta, \delta].$$

This implies that Φ is side-strongly normal at $(0, 0)$. However, Φ is not locally convex-graph at 0, since there exist $x_n = -\frac{1}{n}$, $y_n = -\frac{2}{n}$ and $\lambda = \frac{1}{2}$ such that $\{x_n\}$ and $\{y_n\}$ converge to 0, and

$$\phi\big(\lambda x_n + (1-\lambda)y_n\big) = \frac{3}{2n} - \frac{243}{32n^5} > \frac{3}{2n} - \frac{33}{2n^5} = \lambda\phi(x_n) + (1-\lambda)\phi(y_n).$$

Example 2.2. Let $X = Y = R$. Let $\phi(x) = x^3$ for all $x \in R$. This function is strictly differentiable at 0, and it is easy to show that $\partial\phi(0) = \{0\}$. Let $\Phi(x) = [\phi(x), +\infty)$ for all $x \in R$. Take $(\bar{x}, \bar{y}) = (0, 0)$. By [10, p. 61, Corollary],

$$N\big(\mathrm{Gr}(\Phi), (0,0)\big) = \big\{r\big(x^*, -1\big)\colon\ x^* \in \partial\phi(0),\ r \geqslant 0\big\} = \big\{r(0, -1)\colon\ r \geqslant 0\big\}.$$

Given $\varepsilon > 0$. Take $\delta = \sqrt{\varepsilon}$. Then for all $x \in 0 + \delta B_X = [-\delta, \delta]$, $y = x^3 + h \in \Phi(x)$ $(h \geqslant 0)$ and $r(0, -1) \in N(\mathrm{Gr}(\Phi), (0,0)) \cap (B_{X^*} \times B_{Y^*}) = \{r(0, -1)\colon\ 0 \leqslant r \leqslant 1\}$, we have

$$\langle 0, x - 0\rangle + \langle -r, y - 0\rangle = -r\big(x^3 + h\big) \leqslant -rx^3 \leqslant |x|^3.$$

Noting that

$$|x| \leqslant \delta \quad \Rightarrow \quad |x|^2 \leqslant \varepsilon \quad \Rightarrow \quad |x|^3 \leqslant \varepsilon|x| \quad \Rightarrow \quad |x|^3 \leqslant \varepsilon\big(|x| + |x^3 + h|\big),$$

it follows that

$$\langle 0, x - 0\rangle + \langle -r, y - 0\rangle \leqslant \varepsilon\big(|x| + |x^3 + h|\big) = \varepsilon\big(|x - 0| + |y - 0|\big) \quad \text{for all } x \in [-\delta, \delta].$$

This implies that Φ is strongly normal at $(0, 0)$. However, Φ is not side-strongly normal at $(0, 0)$, since there exist $\varepsilon_0 = \frac{1}{2}$, $(-\frac{1}{n}, -\frac{1}{n^3}) \in \mathrm{Gr}(\Phi)$ and $(0, -1) \in N(\mathrm{Gr}(\Phi), (0,0)) \cap (B_{X^*} \times B_{Y^*})$ such that

$$\left\langle 0, -\frac{1}{n} - 0\right\rangle + \left\langle -1, -\frac{1}{n^3} - 0\right\rangle > \varepsilon_0\left|-\frac{1}{n^3} - 0\right|$$

for all n.

For $(x, y) \in \mathrm{Gr}(\Phi)$, let $D^*\Phi(x, y) : Y^* \to 2^{X^*}$ denote the coderivative of Φ at (x, y), that is,

$$D^*\Phi(x, y)\big(y^*\big) := \big\{x^* \in X^*\colon\ \big(x^*, -y^*\big) \in N\big(\mathrm{Gr}(\Phi), (x, y)\big)\big\} \quad \text{for all } y^* \in Y^*.$$

The very notion of coderivative for multifunctions, regardless of the normal cone used, was first introduced by Mordukhovich in [21]. We refer the readers to his recent book [22] for more details. The coderivative here is defined by Clarke's normal cone, which differs from Mordukhovich's. In the special case when Φ is given by

$$\Phi(x) = \big[f(x), +\infty\big) \quad \text{for all } x \in X,$$

where $f : X \to R \cup \{+\infty\}$ is a proper lower semicontinuous function, Clarke's subdifferential $\partial f(x)$ and the associated coderivative $D^*\Phi(\bar{x}, f(\bar{x})) : R \to 2^{X^*}$ are related [10, p. 61] by

$$\partial f(x) = D^*\Phi\big(x, f(x)\big)(1).$$

Let $C \subset Y$ be a closed convex pointed cone. The positive dual cone C^+ of C is defined by

$$C^+ = \big\{y^* \in Y^*\colon \big\langle y^*, c\big\rangle \geqslant 0 \text{ for all } c \in C\big\}.$$

Recall that C is said to have a bounded base if there exists a bounded convex subset Θ of C such that $C = \{t\theta\colon t \geqslant 0$ and $\theta \in \Theta\}$ and $0 \notin \mathrm{cl}(\Theta)$. It is known that C has a bounded base if and only if $\mathrm{int}(C^+) \neq \emptyset$.

3. Main results

In this section, we always assume that X, Y are Banach spaces, the ordering cone $C \subset Y$ is a closed convex pointed cone and $\Phi : X \to 2^Y$ is a multifunction. For convenience we use the sum norm $\|(x, y)\| = \|x\| + \|y\|$ on the product space $X \times Y$.

The following Theorems 3.1 and 3.2 provide Lagrange multiplier rules for the constrained multiobjective optimization problem (1.2).

Theorem 3.1. *Let Φ be a closed-graph multifunction and Ω be a closed subset of X. Let $(\bar{x}, \bar{y})$ be a local super efficient solution of the constrained multiobjective optimization problem* (1.2). *Suppose that one of the following conditions holds*:

(i) $(H(\Omega, \bar{x}) \times Y) \cap T(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})) \neq \emptyset$;
(ii) $(T(\Omega, \bar{x}) \times Y) \cap H(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})) \neq \emptyset$.

Then for any $b^* \in B_{Y^*}$, there exists $c^* \in C^+$ with $\|c^*\| \leqslant M$ such that

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^* - b^*\big) + N(\Omega, \bar{x}),$$

where $M > 0$ is a constant independent of $b^* \in B_{Y^*}$.

Proof. By the assumption there exists $\delta > 0$ such that $\bar{y} \in SE(\Phi[(\bar{x} + \delta B_X) \cap \Omega], C)$. Then there exists a constant $M > 0$ such that

$$\|y - \bar{y}\| \leqslant M\|\tilde{y}\|$$

for all $y \in \Phi[(\bar{x} + \delta B_X) \cap \Omega]$ and $\tilde{y} \in y - \bar{y} + C$, that is,

$$\|y - \bar{y}\| \leqslant Md(y, \bar{y} - C) \quad \text{for all } y \in \Phi\big[(\bar{x} + \delta B_X) \cap \Omega\big],$$

where $d(y, \bar{y} - C)$ denotes the distance from y to $\bar{y} - C$. Let b^* be an arbitrary element in B_{Y^*}. Then

$$\big\langle b^*, y - \bar{y}\big\rangle \leqslant \|y - \bar{y}\| \leqslant Md(y, \bar{y} - C) \quad \text{for all } y \in \Phi\big[(\bar{x} + \delta B_X) \cap \Omega\big].$$

Letting

$$A_1 = \big\{(x, y) \in X \times Y\colon x \in \bar{x} + \delta B_X,\ y \in \Phi(x)\big\},$$
$$A_2 = \big\{(x, y) \in X \times Y\colon x \in \Omega,\ y \in Y\big\},$$

and letting

$$f(x, y) = -\big\langle b^*, y - \bar{y}\big\rangle + Md(y, \bar{y} - C) \quad \text{for all } (x, y) \in X \times Y,$$

it follows that f attains a local minimum over $A_1 \cap A_2$ at $(\bar{x}, \bar{y})$. This and Proposition 2.1 imply that

$$(0,0) \in \partial f(\bar{x}, \bar{y}) + N\big(A_1 \cap A_2, (\bar{x}, \bar{y})\big).$$

Noting that $-\langle b^*, \cdot - \bar{y}\rangle$ and $d(\cdot, \bar{y} - C)$ are Lipschitz near $(\bar{x}, \bar{y}) \in X \times Y$, it follows from [10, p. 39, Corollary 2] that

$$\partial f(\bar{x}, \bar{y}) \subset \partial\big(-\langle b^*, \bar{y} - \bar{y}\rangle\big) + M\partial d(\bar{y}, \bar{y} - C) = \big(0, -b^*\big) + M\partial d(\bar{y}, \bar{y} - C) \subset \big(0, -b^*\big) + M\big(0, C^+ \cap B_{Y^*}\big).$$

Therefore, there exists $c^* \in C^+$ with $\|c^*\| \leqslant M$ such that

$$\big(0, b^* - c^*\big) \in N\big(A_1 \cap A_2, (\bar{x}, \bar{y})\big). \tag{3.1}$$

Suppose that either (i) or (ii) holds. Without loss of generality, we assume that (i) holds. Since

$$T\big(A_1, (\bar{x}, \bar{y})\big) = \big\{(u, v) \in X \times Y\colon\ (u, v) \in T\big(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})\big)\big\},$$
$$H\big(A_2, (\bar{x}, \bar{y})\big) = \big\{(u, v) \in X \times Y\colon\ u \in H(\Omega, \bar{x}),\ v \in Y\big\},$$

we have

$$T\big(A_1, (\bar{x}, \bar{y})\big) \cap H\big(A_2, (\bar{x}, \bar{y})\big) \neq \emptyset.$$

By Proposition 2.2,

$$N\big(A_1 \cap A_2, (\bar{x}, \bar{y})\big) \subset N\big(A_1, (\bar{x}, \bar{y})\big) + N\big(A_2, (\bar{x}, \bar{y})\big).$$

It follows from (3.1) that

$$\big(0, b^* - c^*\big) \in N\big(A_1, (\bar{x}, \bar{y})\big) + N\big(A_2, (\bar{x}, \bar{y})\big). \tag{3.2}$$

Since

$$N\big(A_1, (\bar{x}, \bar{y})\big) = \big\{\big(x^*, y^*\big) \in X^* \times Y^*\colon\ \big(x^*, y^*\big) \in N\big(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})\big)\big\},$$
$$N\big(A_2, (\bar{x}, \bar{y})\big) = \big\{\big(x^*, 0\big) \in X^* \times Y^*\colon\ x^* \in N(\Omega, \bar{x})\big\},$$

it follows from (3.2) that there exist $(x_1^*, y_1^*) \in N(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y}))$ and $x_2^* \in N(\Omega, \bar{x})$ such that

$$\big(0, b^* - c^*\big) = \big(x_1^*, y_1^*\big) + \big(x_2^*, 0\big).$$

This implies that $0 = x_1^* + x_2^*$, $b^* - c^* = y_1^*$. Therefore,

$$0 = x_1^* + x_2^* \in D^*\Phi(\bar{x}, \bar{y})\big(-y_1^*\big) + N(\Omega, \bar{x}) = D^*\Phi(\bar{x}, \bar{y})\big(c^* - b^*\big) + N(\Omega, \bar{x}). \qquad \square$$

Remark. The following example shows that, dropping the conditions (i) and (ii), the conclusion of Theorem 3.1 may not be true even Φ is a convex-graph multifunction and Ω is a convex set.

Example 3.1. Let $X = Y = R$. Let $C = [0, +\infty)$. let $\Omega = [0, +\infty)$. Let $\Phi : R \to 2^R$ be defined by

$$\Phi(x) = \begin{cases} [-\sqrt{1-(1+x)^2}, \sqrt{1-(1+x)^2}\,], & x \in [-2, 0], \\ \emptyset, & \text{otherwise}. \end{cases}$$

Then $\mathrm{Gr}(\Phi) = \{(x, y)\colon\ -2 \leqslant x \leqslant 0,\ (x+1)^2 + y^2 \leqslant 1\}$. Clearly, Φ is a multifunction with a closed and convex graph. Take $(\bar{x}, \bar{y}) = (0, 0)$. It is easy to see that $(0, 0)$ is a local super efficient solution of the problem (1.2) since for all $\delta > 0$, $\Phi(\Omega \cap (0 + \delta B_X)) = \{0\}$. However, for $b^* = -1 \in B_{Y^*} = [-1, 1]$, there does not exist $c^* \in C^+ = [0, +\infty)$ such that

$$0 \in D^*\Phi(0, 0)\big(c^* - b^*\big) + N(\Omega, 0),$$

since $N(\mathrm{Gr}(\Phi), (0, 0)) = \{(x, 0)\colon\ x \geqslant 0\}$ and $N(\Omega, 0) = (-\infty, 0]$.

Remark. Even in the convex case, condition (i) or (ii) in Theorem 3.1 is not necessary for the Lagrange rule to be true. See the following example.

Example 3.2. Let $X = Y = R$. let $\Omega = \{0\}$. Let $C = [0, +\infty)$. Let $\Phi(0) = 0$ and $\Phi(x) = \emptyset$ for $x \in R \setminus \{0\}$. Then $\mathrm{Gr}(\Phi) = \{(0,0)\}$. Clearly, Φ is a multifunction with a closed and convex graph. Take $(\bar{x}, \bar{y}) = (0,0)$. It is easy to see that $(0,0)$ is a local super efficient solution of the problem (1.2). By the definition of Clarke's tangent cone, we have $T(\mathrm{Gr}(\Phi), (0,0)) = \{(0,0)\}$ and $T(\Omega, 0) = \{0\}$. Since $\mathrm{int}(T(\mathrm{Gr}(\Phi), (0,0))) = \emptyset$ and $\mathrm{int}(T(\Omega, 0)) = \emptyset$, by [10, p. 57, Theorem 2.4.8], $H(\Omega, 0) = \emptyset$ and $H(\mathrm{Gr}(\Phi), (0,0)) = \emptyset$. It follows that

$$\big(H(\Omega, 0) \times Y\big) \cap T\big(\mathrm{Gr}(\Phi), (0,0)\big) = \emptyset$$

and

$$\big(T(\Omega, 0) \times Y\big) \cap H\big(\mathrm{Gr}(\Phi), (0,0)\big) = \emptyset.$$

The conditions (i) and (ii) in Theorem 3.1 are not satisfied. However, For each $b^* \in B_{Y^*} = [-1, 1]$, there exists $c^* = 3 \in C^+ = [0, +\infty)$ such that

$$0 \in D^*\Phi(0,0)\big(3 - b^*\big) + N(\Omega, 0).$$

Since $N(\mathrm{Gr}(\Phi), (0,0)) = R \times R$ and $N(\Omega, 0) = R$. The conclusion of Theorem 3.1 holds true.

Theorem 3.2. *Let Φ be a closed-graph multifunction and Ω be a closed subset of X. Let $C \subset Y$ be a closed convex cone with a bounded base. Let $(\bar{x}, \bar{y})$ be a local super efficient solution of the constrained multiobjective optimization problem* (1.2)*. Suppose that one of the following conditions holds*:

(i) $(H(\Omega, \bar{x}) \times Y) \cap T(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})) \neq \emptyset$;
(ii) $(T(\Omega, \bar{x}) \times Y) \cap H(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})) \neq \emptyset$.

Then there exists $c^* \in \mathrm{int}(C^+)$ with $\|c^*\| = 1$ such that

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^*\big) + N(\Omega, \bar{x}).$$

Proof. By Theorem 3.1, for each $b^* \in B_{Y^*}$, there exists $\tilde{c}^* \in C^+$ such that

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(\tilde{c}^* - b^*\big) + N(\Omega, \bar{x}). \tag{3.3}$$

Since C has a bounded base, $\mathrm{int}(C^+) \neq \emptyset$. Take $c_1^* \in \mathrm{int}(C^+) \cap B_{Y^*}$. Clearly, $-c_1^* \in (-\mathrm{int}(C^+)) \cap B_{Y^*}$. By (3.3), there exists $c_2^* \in C^+$ such that

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c_2^* + c_1^*\big) + N(\Omega, \bar{x}).$$

Since C^+ is a convex cone, we get $c_2^* + c_1^* \in C^+ + \mathrm{int}(C^+) \subset \mathrm{int}(C^+)$ and $\frac{c_2^* + c_1^*}{\|c_2^* + c_1^*\|} \in \mathrm{int}(C^+)$. Letting $c^* = \frac{c_2^* + c_1^*}{\|c_2^* + c_1^*\|}$, since $D^*\Phi(\bar{x}, \bar{y})(\cdot)$ is positively homogeneous and $N(\Omega, \bar{x})$ is a cone, one has

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^*\big) + N(\Omega, \bar{x}). \qquad \square$$

The following Theorems 3.3, 3.4 and 3.5 provide sufficient conditions for local super efficient solutions.

Theorem 3.3. *Let Φ be a closed-graph multifunction and Ω be a closed convex subset of X. Let $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$. Suppose that Φ is side-strongly normal at $(\bar{x}, \bar{y})$ and that for any $b^* \in B_{Y^*}$, there exists $c^* \in C^+$ with $\|c^*\| \leqslant M$ such that*

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^* - b^*\big) + N(\Omega, \bar{x}) \cap M B_{X^*}, \tag{3.4}$$

where $M > 0$ is a constant independent of $b^* \in B_{Y^*}$. Then $(\bar{x}, \bar{y})$ is a local super efficient solution of the constrained multiobjective optimization problem (1.2)*.*

Proof. Since Φ is side-strongly normal at $(\bar{x}, \bar{y})$, there exists $\delta > 0$ such that

$$\big\langle x^*, u - \bar{x}\big\rangle + \big\langle y^*, v - \bar{y}\big\rangle \leqslant \frac{1}{4M+2}\|v - \bar{y}\| \tag{3.5}$$

for all $(x^*, y^*) \in N(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})) \cap (B_{X^*} \times B_{Y^*})$ and all $u \in \bar{x} + \delta B_X$ and $v \in \Phi(u)$. Let $y \in \Phi[(\bar{x} + \delta B_X) \cap \Omega]$ and $w \in Y$ with $y - \bar{y} \leqslant_C w$. Then there exist $x \in (\bar{x} + \delta B_X) \cap \Omega$ and $c \in C$ such that $y \in \Phi(x)$ and $y - \bar{y} = w - c$. Take $b^* \in B_{Y^*}$ such that

$$\langle b^*, y - \bar{y} \rangle = \|y - \bar{y}\|. \tag{3.6}$$

By (3.4), there exists $c^* \in C^+$ with $\|c^*\| \leqslant M$ such that

$$0 \in D^*\Phi(\bar{x}, \bar{y})(c^* - b^*) + N(\Omega, \bar{x}) \cap MB_{X^*}.$$

Then there exists $x_1^* \in X^*$ such that

$$x_1^* \in D^*\Phi(\bar{x}, \bar{y})(c^* - b^*)$$

and

$$-x_1^* \in N(\Omega, \bar{x}) \cap MB_{X^*}.$$

Clearly, $\|b^* - c^*\| + \|x_1^*\| \leqslant \|b^*\| + \|c^*\| + \|x_1^*\| \leqslant 1 + 2M$. It follows that $(\frac{x_1^*}{2M+1}, \frac{b^*-c^*}{2M+1}) \in N(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})) \cap (B_{X^*} \times B_{Y^*})$. By (3.5),

$$\langle x_1^*, x - \bar{x} \rangle + \langle b^* - c^*, y - \bar{y} \rangle \leqslant \frac{1}{2}\|y - \bar{y}\|. \tag{3.7}$$

By the convexity of Ω, we have

$$\langle -x_1^*, x - \bar{x} \rangle \leqslant 0. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\langle b^* - c^*, y - \bar{y} \rangle \leqslant \frac{1}{2}\|y - \bar{y}\|. \tag{3.9}$$

This and (3.6) imply that

$$\begin{aligned} \|y - \bar{y}\| &\leqslant \frac{1}{2}\|y - \bar{y}\| + \langle c^*, y - \bar{y} \rangle = \frac{1}{2}\|y - \bar{y}\| + \langle c^*, w - c \rangle \leqslant \frac{1}{2}\|y - \bar{y}\| + \langle c^*, w \rangle \\ &\leqslant \frac{1}{2}\|y - \bar{y}\| + \|c^*\| \cdot \|w\| \leqslant \frac{1}{2}\|y - \bar{y}\| + M\|w\|, \end{aligned}$$

that is,

$$\|y - \bar{y}\| \leqslant 2M\|w\|.$$

This implies that $\bar{y} \in SE(\Phi[(\bar{x} + \delta B_X) \cap \Omega], C)$. Hence $(\bar{x}, \bar{y})$ is a local super efficient solution of the problem (1.2). □

The following example shows that, dropping the assumption that Φ is side-strongly normal at $(\bar{x}, \bar{y})$, the conclusion of Theorem 3.3 may not be true even in finite dimensional space.

Example 3.3. Let $X = Y = R$, $C = [0, +\infty)$ and $\Omega = R$. Let $\Phi : R \to 2^R$ be defined by

$$\Phi(x) = [x^3, +\infty) \quad \text{for all } x \in R.$$

Take $(\bar{x}, \bar{y}) = (0, 0) \in \mathrm{Gr}(\Phi)$. It is easy to verify that $N(\mathrm{Gr}(\Phi), (0, 0)) = \{(0, -t) \in R \times R\colon t \geqslant 0\}$ and $N(\Omega, 0) = \{0\}$. For each $b^* \in B_{Y^*} = [-1, 1]$, there exists $c^* = 3 \in C^+$ such that

$$0 \in D^*\Phi(0, 0)(3 - b^*) + N(\Omega, 0) \cap B_{X^*}.$$

However, $(0, 0)$ is not a local super efficient solution of the problem (1.2). Since for all $\delta > 0$,

$$\operatorname{cl}\operatorname{cone}(\Phi[(0 + \delta B_X) \cap \Omega] - 0) \cap (B_Y - C) = (-\infty, +\infty) \cap (-\infty, 1] = (-\infty, 1].$$

Theorem 3.4. *Let Φ be a multifunction with a closed and convex graph, Ω be a closed convex subset of X and $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$. Suppose that for any $b^* \in B_{Y^*}$, there exists $c^* \in C^+$ with $\|c^*\| \leqslant M$ such that*

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^* - b^*\big) + N(\Omega, \bar{x}), \tag{3.10}$$

where $M > 0$ is a constant independent of $b^* \in B_{Y^*}$. Then $(\bar{x}, \bar{y})$ is a local super efficient solution of the constrained multiobjective optimization problem (1.2).

Proof. Let $y \in \Phi(\Omega)$ and $w \in Y$ with $y - \bar{y} \leqslant_C w$. Noting that Φ is a convex-graph multifunction, then the right-hand side of (3.7) in the proof Theorem 3.3 can be replaced by 0, and then (3.9) becomes

$$\big\langle b^* - c^*, y - \bar{y}\big\rangle \leqslant 0.$$

This justifies the super efficiency of $(\bar{x}, \bar{y})$ due to

$$\|y - \bar{y}\| = \big\langle b^*, y - \bar{y}\big\rangle \leqslant \big\langle c^*, y - \bar{y}\big\rangle \leqslant \big\langle c^*, w\big\rangle \leqslant \big\|c^*\big\| \cdot \|w\| \leqslant M\|w\|. \qquad \Box$$

Remark. In comparison with Theorems 3.3, 3.4 requires the convexity assumption on Φ instead of the side-strong normality, but the condition (3.10) replaces the stronger condition (3.4).

Theorem 3.5. *Let Φ be a closed-graph multifunction and Ω be a closed convex subset of X. Let $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$. Suppose that Φ is side-strongly normal at $(\bar{x}, \bar{y})$ and that there exists $c^* \in \mathrm{int}(C^+)$ with $\|c^*\| = 1$ such that*

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^*\big) + N(\Omega, \bar{x}).$$

Then $(\bar{x}, \bar{y})$ is a local super efficient solution of the constrained multiobjective optimization problem (1.2).

Proof. By the assumption, there exists $c^* \in \mathrm{int}(C^+)$ with $\|c^*\| = 1$ such that

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^*\big) + N(\Omega, \bar{x}).$$

Then there exists $x^* \in X^*$ such that

$$x^* \in D^*\Phi(\bar{x}, \bar{y})\big(c^*\big) \quad \text{and} \quad -x^* \in N(\Omega, \bar{x}).$$

Let $M := \|x^*\|$. Since $c^* \in \mathrm{int}(C^+)$, there exists $\theta > 0$ such that $c^* + \theta B_{Y^*} \subset C^+$. Let b^* be an arbitrary element in B_{Y^*}. Then there exists $\tilde{c}^* \in C^+$ such that $c^* + \theta b^* = \tilde{c}^*$. Clearly, $\frac{\tilde{c}^*}{\theta} \in C^+$, $\frac{\tilde{c}^*}{\theta} - b^* = \frac{c^*}{\theta}$ and

$$\left\|\frac{\tilde{c}^*}{\theta}\right\| \leqslant 1 + \frac{1}{\theta} < \frac{1 + \theta + M}{\theta}.$$

It follows that

$$\begin{aligned}
0 &= \frac{1}{\theta}x^* + \frac{1}{\theta}\big(-x^*\big) \\
&\in \frac{1}{\theta}D^*\Phi(\bar{x}, \bar{y})\big(c^*\big) + N(\Omega, \bar{x}) \cap \frac{M}{\theta}B_{X^*} \\
&= D^*\Phi(\bar{x}, \bar{y})\left(\frac{c^*}{\theta}\right) + N(\Omega, \bar{x}) \cap \frac{M}{\theta}B_{X^*} \\
&\subset D^*\Phi(\bar{x}, \bar{y})\left(\frac{\tilde{c}^*}{\theta} - b^*\right) + N(\Omega, \bar{x}) \cap \frac{1 + \theta + M}{\theta}B_{X^*}.
\end{aligned}$$

By Theorem 3.3, $(\bar{x}, \bar{y})$ is a local super efficient solution of the constrained multiobjective optimization problem (1.2). $\Box$

Remark. In comparison with Theorems 3.3 and 3.4, Theorem 3.5 only requires that there exists one element $c^* \in \mathrm{int}(C^+)$ such that $0 \in D^*\Phi(\bar{x}, \bar{y})(c^*) + N(\Omega, \bar{x})$. However, $\mathrm{int}(C^+) \neq \emptyset$ is equivalent to C having a bounded base. Theorems 3.3 and 3.4 do not require that C has a bounded base, but require that for each $b^* \in B_{Y^*}$, there exists $\tilde{c}^* \in C^+$ such that (3.4) and (3.10) hold, respectively.

In the special case when $\Omega = X$, (1.2) is reduced to the following problem:

$$C - \min \Phi(x), \tag{3.11}$$

and $N(\Omega, x) = \{0\}$ for all $x \in X$. Thus Corollaries 3.1 and 3.2 are immediate consequences of Theorems 3.1 and 3.2.

Corollary 3.1. *Let Φ be a closed-graph multifunction. Let $(\bar{x}, \bar{y})$ be a local super efficient solution of the unconstrained multiobjective optimization problem* (3.11). *Then for any $b^* \in B_{Y^*}$, there exists $c^* \in C^+$ with $\|c^*\| \leqslant M$ such that*

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^* - b^*\big),$$

where $M > 0$ is a constant independent of $b^* \in B_{Y^*}$.

Corollary 3.2. *Let Φ be a closed-graph multifunction. Let $C \subset Y$ be a closed convex cone with a bounded base. Let $(\bar{x}, \bar{y})$ be a local super efficient solution of the unconstrained multiobjective optimization problem* (3.11). *Then there exists $c^* \in \operatorname{int}(C^+)$ with $\|c^*\| = 1$ such that*

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^*\big).$$

Remark. Let X be a Banach space, and $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous function. The following result is well-known as the generalized Fermat's rule:

$$f \text{ attains a local minimum at } x \quad \Rightarrow \quad 0 \in \hat{\partial} f(x) \subset \partial_M f(x) \subset \partial f(x),$$

where $\hat{\partial} f(\bar{x})$ and $\partial_M f(\bar{x})$ are the Frechet and Mordukhovich subdifferentials of f at $\bar{x}$, respectively (see [22]). Recently, Zheng and Ng [17] gave Fermat's rule for multifunctions for Pareto efficiency. Thus Corollaries 3.1 and 3.2 can be regarded as Fermat's rule for multifunctions for super efficiency.

The following two corollaries are immediate consequences of Theorems 3.3 and 3.5.

Corollary 3.3. *Let Φ be a closed-graph multifunction and be side-strongly normal at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$. Suppose that for any $b^* \in B_{Y^*}$, there exists $c^* \in C^+$ with $\|c^*\| \leqslant M$ such that*

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^* - b^*\big),$$

where $M > 0$ is a constant independent of $b^* \in B_{Y^*}$. Then $(\bar{x}, \bar{y})$ is a local super efficient solution of the unconstrained multiobjective optimization problem (3.11).

Corollary 3.4. *Let Φ be a closed-graph multifunction and be side-strongly normal at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$. Suppose that there exists $c^* \in \operatorname{int}(C^+)$ with $\|c^*\| = 1$ such that*

$$0 \in D^*\Phi(\bar{x}, \bar{y})\big(c^*\big).$$

Then $(\bar{x}, \bar{y})$ is a local super efficient solution of the unconstrained multiobjective optimization problem (3.11).

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